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AN ANALYTICAL STUDY OF THE DIFFERENTIAL FRICTIONAL EFFECT ON VORTEX MOVEMENT

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ABSTRACT

With a view to gaining insight into the role of differential friction in the movement of tropical cyclones, an analytical solution of the barotropic vorticity equation which included a frictionally induced vertical velocity term was obtained in the form of a Taylor's series in time. An axisymmetric vortex of maximum radius 1000 km having a peak tangential motion of 30 m sec^{-1} about 450 km from the center was studied. If the eastern (western) half of such a vortex lies over a rough surface with the other half over a relatively smooth surface, the familiar west-northwestward movement of the vortex due to β-effect is increased (decreased) by the differential friction. If only the underlying surface of the northern (southern) half of the vortex is rough, a 46-km, nearly westward (a 40-km northwestward) movement is produced by the β- and differential frictional effects over a 6-hr period. These movements, however, are nonlinear in time. The dependence of the vortex movement on the strength and size of the vortex circulation and on the frictional stress is described.

1. INTRODUCTION

The barotropic vorticity equation has been integrated numerically (for example, Kasahara 1957) to predict the displacement of tropical cyclones. The same equation was utilized to investigate analytically the influence of the variation of the Coriolis parameter (β-effect) on vortex movement (Adem 1956). Although the induced west-northwestward motion due to this effect is commonly smaller than the displacement due to large-scale flow, the movement due to the β-effect, according to both Adem and Kasahara, may not be ignored when the large-scale flow is weak and when predictions exceeding 12 hr are required.

In forecast studies of tropical cyclone movement, a situation is sometimes encountered wherein a large part of the vortex circulation lies over land with the other part over water. The movement of such a vortex is determined in simple atmospheric prediction models by three factors: the large-scale flow, the β-effect, and the differential friction (called DF in this study). Adem (1956) and Adem and Lezama (1960) studied analytically the influence of the first two factors on vortex movement, while Rao (1969) investigated the role of DF on hurricane movement, utilizing real data and a barotropic model. No analytical study, however, seems to have been made on the role of DF in the movement of tropical cyclones.

The aim of this study is to make an analytical estimate of the effect of DF on the movement of a symmetric vortex. The attempt should be regarded as a straightforward extension of Adem's (1956) work except that, in this investigation, friction was included in the barotropic vorticity equation, whereas Adem did not have such a factor.

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2. BASIC METHOD

With suitable boundary conditions and with the standard notation, a solution to the Poisson equation in polar coordinates

$$\nabla^2\phi = F\ (R)\ \cos m\theta \tag{1}$$

may be written as

$$\phi = R^m \cos m\theta (2m)^{-1} \int F(R) R^{-m+1} dR - \cos m\theta (2mR^m)^{-1} \int F(R) R^{m+1} dR. \tag{2}$$

In equation (2), $F(R)$ is a specified function of position and ϕ is the unknown function.

A forecast stream function, ψ_f, may be expanded about an initial stream function, ψ, in terms of a Taylor's series in time, thus:

$$\psi_f = \psi + \psi_t \Delta T + \psi_{tt} (\Delta T)^2/2! + \ \ldots\ . \tag{3}$$

The quantity ψ_f can be estimated if ψ, ψ_t, ψ_{tt}, etc. are known. In the above, $\psi_t = \partial\psi/\partial t$ etc. The barotropic vorticity equation with friction (Charney and Eliassen 1949) may be written as

$$\nabla^2\psi_t = J(\nabla^2\psi + f, \psi) - \chi\nabla^2\psi - \psi\nabla^2\chi \tag{4}$$

from which ψ_t can be estimated by applying (2), provided the right side of (4) can be expressed as a linear combination of circular functions. The quantity χ is a measure of the frictional influence and, following Charney and Eliassen (1949), may be written as $\chi = 0.4F$ where $F = \sin 2\alpha\sqrt{Kf}/\sqrt{2}H$.

In the above, the number 0.4 represents the ratio of the strength of the surface winds to that of the upper air winds, say at 500 mb; α, the angle between the surface winds and surface isobars; K, the coefficient of eddy viscosity; and H, the height of a homogeneous atmosphere. Charney and Eliassen obtained a value of 0.8×10^{-6} sec^{-1} for χ. In the following, however, χ is set equal to $Ta \cos \theta$, where T has the constant value, 0.5×10^{-5} sec^{-1} and a is a dimensionless variable between 0 to 1 given at intervals of 0.1. Close to the center, where $a=0.1$, $\chi=0.5\times10^{-6}$ sec^{-1} and this compares favorably with the value used by the above-mentioned investigators. In this formulation, friction was regarded as an energy source on one side of the vortex and as sink on the other side. Such a formulation does not reproduce the dissipative property of friction. However, it was used because of its simplicity; and because the effect of DF on vortex movement was being calculated over a period of only 6 hr.

Differentiation of equation (4) with respect to t yields

$$\nabla^2\psi_{tt}=J(\nabla^2\psi_t,\psi)+J(\nabla^2\psi+f,\psi_t)-\chi\nabla^2\psi_t-\psi_t\nabla^2\chi \tag{5}$$

from which ψ_{tt} in turn can be estimated, following the procedure given above for ψ_t. An extension of this procedure also yields higher temporal derivatives of ψ.

3. THE DATA

Consider an axisymmetric vortex given by

$$\psi=\psi_0(1-a^2)^3 \tag{6}$$

where $r/r_0=a$ and $0 \leq a \leq 1$. The quantity ψ_0 may be found from the following equation:

$$\psi_0=\psi_{r,max}r_0/(-6a_m+12a_m^3-6a_m^5).$$

Here, the maximum tangential motion, $\psi_{r,max}$, of the vortex was reached at a_m. For the ψ-profile given by equation (6), $a_m=\sqrt{0.2}$. In the following study, case A with $r_0=1000$ km and $\psi_{r,max}=30$ m sec^{-1}, case B with $r_0=1000$ km and $\psi_{r,max}=40$ m sec^{-1}, case C with $r_0=500$ km and $\psi_{r,max}=30$ m sec^{-1}, and case D with $r_0=500$ km and $\psi_{r,max}=40$ m sec^{-1} were considered. The tangential motion, ψ_r, and the relative vorticity $\nabla^2\psi$, of the flow are easily deducible from equation (6) for these cases. The frictionally induced vertical velocities, ω's, could be estimated by dividing the quantity $Ta \cos\theta\nabla^2\psi$ by the quantity (f_0/800 mb), where f_0 is the Coriolis parameter, with a constant value of 0.6×10^{-4} sec^{-1}. Table 1 gives the ψ_r's, and ω's, over the x-axis for case A. The vertical motions close to the center are of major consequence in the calculation of the movement of the vortex. These are underestimated when compared to the mean values of low-level vertical motion given by Izawa (1964). Hence it is believed that the frictional stress was not overestimated in this study.

A constant value of β equal to 2.07×10^{-13} cm^{-1} sec^{-1}, characteristic for latitude 25° N., was used in the calculation. The following boundary conditions were utilized: $\psi_t=0$ at $a=0$ and at $a=a_1$, and $\psi_{tt}=0$ at $a=0$ and at $a=1$. The quantity a_1 was chosen to have the value 1.4 so that the calculation would yield a westward

TABLE 1.—*Tangential velocities, ψ_r's, in units of m sec^{-1} and frictionally induced vertical velocities, ω's, in units of 10^{-3} mb sec^{-1} over the x-axis for case* A. *The quantity (a) represents a dimensionless radius*

a	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
ψ_r	0	10.2	19.3	26.0	29.5	29.4	25.7	19.0	10.8	3.4	0
ω	0	−1.3	−2.3	−2.7	−2.4	−1.3	0.4	2.3	3.6	3.4	0

TABLE 2.—*Numerical estimates on the x-axis of $\psi_{t,\beta}$ and $\psi_{t,Fr}$, in units of 10^5 cm^2 sec^{-2}, and numerical estimates on the y-axis of $\psi_{tt,\beta}$ and $\psi_{tt,Fr}$, in units of cm^2 sec^{-3} for cases* A *and* D. *Eastern half of the vortex was assumed to be lying over a rough surface*

a	A(r_0=1000 km; $\psi_{r,max}$=30 m sec^{-1})				D(r_0=500 km; $\psi_{r,max}$=40 m sec^{-1})			
	$\psi_{t,\beta}$	$\psi_{t,Fr}$	$\psi_{tt,\beta}$	$\psi_{tt,Fr}$	$\psi_{t,\beta}$	$\psi_{t,Fr}$	$\psi_{tt,\beta}$	$\psi_{tt,Fr}$
0.1	1.7	0.7	−4.9	−3.8	0.5	0.5	−4.3	−6.7
.3	4.6	2.0	−12.6	−10.5	1.5	1.3	−11.2	−18.8
.5	6.0	2.8	−14.5	−13.5	2.0	1.8	−12.9	−24.1
.7	5.7	3.4	−9.2	−9.8	1.9	2.2	−8.1	−17.5
.9	4.7	4.7	−1.5	−1.8	1.5	3.1	−1.3	−3.3

movement (due to β) which was in agreement with that for a similar vortex given by Adem (1956).

4. SOLUTION

Equation (4) may be written as

$$\nabla^2\psi_t=-\beta\psi_r \cos\theta-Ta \cos\theta\nabla^2\psi. \tag{7}$$

By using (6) and (2), a solution of equation (7) due to the β-effect alone may be written as:

$$\psi_{t,\beta}=x_{11}C(x_{11}A-a^2/2+a^4/3-a^6/12)a \cos\theta \tag{8}$$

and another solution due to the DF effect only may be written as

$$\psi_{t,Fr}=x_{11}D(x_{11}B-a^2/2+2a^4/3-a^6/4)a \cos\theta. \tag{9}$$

In the above,

$x_{11A}=-(-a_1^2/2+a_1^4/3-a_1^6/12)$; $x_{11B}=-(-a_1^2/2+2a_1^4/3-a_1^6/4)$;
$x_{11C}=-3\psi_0r_0\beta/2$; and $x_{11D}=-3\psi_0T$.

Numerical estimates of $\psi_{t,\beta}$ and $\psi_{t,Fr}$ are made for cases A, B, C, and D. Data relating to cases A and D only are given in table 2 for $\theta=0$.

Further, equation (5) may be written as

$$\nabla^2\psi_{tt}=F_1+F_2+F_3+F_4$$

where

$$F_1\equiv48\psi_0r_0^{-4}(-a+2a^3-a^5)(x_{12}+3x_{13}a^2+6x_{14}a^4)\sin\theta;$$
$$F_2\equiv-48\psi_0r_0^{-4}(2a-3a^3)(x_{11}+x_{12}a^2+x_{13}a^4+x_{14}a^6)\sin\theta;$$
$$F_3\equiv-\beta r_0^{-1}(x_{11}+3x_{12}a^2+5x_{13}a^4+7x_{14}a^6)\cos^2\theta; \text{ and}$$
$$F_4\equiv-8Tar_0^{-2}(x_{12}a+3x_{13}a^3+6x_{14}a^5)\cos^2\theta.$$

TABLE 3.—*Case A, estimates of F_3 and F_4 for $\theta=0$ and those of F_1, F_2, F_3, and F_4 for $\theta=45°$; units, 10^{-14} sec^{-3}*

	a	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\theta=0$												
	F_3	−0.53	−0.50	−0.43	−0.33	−0.21	−0.11	−0.03	−0.00	−0.01	−0.06	−0.10
	F_4	0	.01	.05	.11	.15	.17	.15	.07	−.02	−.08	−0
$\theta=45°$												
	F_1	0	−2.27	−3.94	−4.59	−4.16	−2.95	−1.52	−0.42	0.05	.04	0
	F_2	0	2.95	5.37	6.87	7.24	6.54	4.99	2.89	0.43	−2.40	−5.86
	F_3	−0.53	−0.25	−0.21	−0.16	−0.10	−0.05	−0.01	−0	−0	−0.03	−0.05
	F_4	0	0	.02	.05	.07	.08	.07	.03	−0.01	−.04	−0

In the above,

$$x_{11}=(x_{11B}x_{11D}+x_{11A}x_{11C});x_{12}=-(x_{11D}+x_{11C})/2;$$
$$x_{13}=(2x_{11D}+x_{11C})/3; \text{ and } x_{14}=-(x_{11D}/4+x_{11C}/12).$$

F_1 and F_2 contain linear combinations of the β- and DF-terms and are both multiplied by $\sin\theta$. On the other hand, F_3 and F_4 contain the products of the β- and DF-terms, and are both multiplied by $\cos^2\theta$. The combination of F_3 and F_4 was smaller than that of F_1 and F_2 except at the center and on the x-axis. This is evident from table 3 where the values of F_1, F_2, F_3, and F_4 were shown. Hence, an approximate solution to (10) may be written as

$$\psi_{tt}=-12\psi_0 r_0^{-2}a\sin\theta(x_{111}+(x_{11}+x_{12}/2)a^2+(x_{13}-x_{11})a^4/2$$
$$+(3x_{14}-2x_{13}-x_{12})a^6/6+x_{14}(a^2/5-1)a^8/2) \quad (11)$$

after ignoring the last two expressions on the right of (10). In the above,

$$x_{111}=-((x_{11}+x_{12}/2)+(x_{13}-x_{11})/2$$
$$+(3x_{14}-2x_{13}-x_{12})/6-2x_{14}/5).$$

Estimates of ψ_t are made for cases A, B, C, and D with $T=0$ and $\beta=0$ separately. Only those pertaining to A and D are shown in table 2 for $\theta=90°$.

In this study, the cubic term ψ_{ttt} was not calculated. Adem (1956) showed that the 12-hr movement arising from this term is only about 10 percent in magnitude of and opposite in direction to that arising from the linear term.

By taking ΔT to be 6 hr, the westward and northward vortex displacements were found from a knowledge of the $\psi+\psi_t\,\Delta T+\psi_{tt}\,\Delta T^2/2$ field close to the center; these are given in table 4. The northward displacement due to the β-effect for case A compares favorably with the displacement (9 km) obtained by Adem.

From table 4 it may be seen that the linear movement due to β and DF are directly proportional to r_0^2 and r_0 respectively. The cause for this may be understood from (8) and (9). The northward movements due to β, however, are dependent both on the intensity and the size of the vortex, while those induced by the DF are dependent only on the intensity of the vortex circulation. This can be explained by appealing to equation (11). When $T=0$, the displacement is proportional to $\psi_{r,max}r_0$. Further, when β vanishes, the displacement becomes proportional to $\psi_{r,max}$.

TABLE 4.—*Six-hour displacement of a vortex in kilometers, due to β- and DF effects making use of the estimates of ψ_t and ψ_{tt}. Units, r_0 in kilometers and $\psi_{r,max}$ in meters per second. The eastern half of the vortex was assumed to be lying over a rough surface*

		A	B	C	D
	r_0	1000	1000	500	500
Westward, linear in time	$\psi_{r,max}$	30	40	30	40
1. due to β		36.4	36.4	9.0	9.0
2. due to DF		15.9	15.9	8.0	8.0
Northward, quadratic in time					
1. due to β		11.0	14.7	5.5	7.3
2. due to DF		8.5	11.4	8.5	11.4

The above calculation was repeated for the situation when the northern half of the vortex rested on a rough surface. A combination of β- and DF-effects yielded the following nearly westward 6-hr displacements, nonlinear in time: case A, 46 km; case B, 48 km; case C, 18 km; and case D, 21 km. If the southern half of the vortex was lying over a rough sruface, the following 6-hr displacements were obtained: case A, 40 km; case B, 40 km; case C, 13 km; and case D, 16 km. Cases A and B were northwestward, and cases C and D were nearly northward.

Synoptic analyses of some tropical cyclones show certain asymmetries of flow. The greater the asymmetry of the flow, the less useful are the present results for the prediction purpose. Using real data, Rao (1969) studied the combined role of the asymmetry of the flow and DF in hurricane movement. According to this study, due to the asymmetry of flow alone the vortex moves away from that half of the cyclone which has stronger tangential winds. The resultant movement, of course, also depends on the β- and DF-effects, the amount of asymmetry, and the large-scale flow.

5. CONCLUSION

It is somewhat difficult to apply these results directly to the problem of predicting hurricane movement, since the formulation of the frictional stress was grossly oversimplified and the asymmetry, which is commonly present in hurricane flows, was ignored. However, for short-range predictions, if no other influences are present, we conclude that if the eastern (western) half of an axisymmetric vortex lies over a rough surface, its west-northwestward movement due to the β-effect is increased (reduced). If the northern (southern) half of a symmetric vortex of 1000-km radius, having a peak tangential motion of 30 m sec^{-1}, lies over a rough surface, the vortex moves nearly westward about 46 km (northwestward, about 40 km) over a period of 6 hr as a consequence of both the β- and DF-effects. These displacements, however, are nonlinear in time. Independent of the intensity of circulation, the linear β- and DF-displacements are respectively proportional to r_0^2 and r_0. The quadratic (in time) displacements due to β, on the other hand, are dependent both on the intensity and size of the circulation, while the corresponding displacements due to DF are proportional to the intensity of circulation only.

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